SOCIETE GENERALE
Cross Asset Research

01 March 2012

Fixed Income

Special comment

The Magpie Yield Curve Model at SG

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Key points:

Magpie is a useful and clear way to see relative value within a set of bonds.

The user might wish to know, but does not need to know, that:

$$r^{\text{Govt}}(t) = y_0^{\text{Govt}} + y_1^{\text{Govt}} e^{-t/z_1} + \ldots + y_5^{\text{Govt}} e^{-t/z_5}$$
$$r^{\text{Repo}}(t) = y_0^{\text{Repo}} + y_1^{\text{Repo}} e^{-t/z_1} + \ldots + y_5^{\text{Repo}} e^{-t/z_5}$$

$$\text{'}z_0\text{'} = +\infty;\ z_1 = 12 \text{ years};\ z_2 = 5 \text{ years};\ z_3 = 2 \text{ years};\ z_4 = 6 \text{ months};\ z_5 = 1 \text{ month}$$

$$y_5^{\text{Govt}} = y_5^{\text{Repo}};\ y_4^{\text{Govt}} = y_4^{\text{Repo}};\ \ldots;\ y_1^{\text{Govt}} = y_1^{\text{Repo}};\ y_0\text{s not so constrained}$$

$$Error = \sum_{\text{Tradeables}} \left(\frac{\text{Ln}\left(\frac{\text{Theoretical}[y_i]}{Market}\right)}{\frac{\partial \text{Theoretical}[y_i]}{\partial y_0} \times \frac{1}{\text{Theoretical}[y_i]}} \right)^2$$

Société Générale has built an implementation of the Magpie Yield Curve model. This model elegantly and parsimoniously captures the essence of a yield curve. Further, the output is elegantly simple, revealing useful truth without false model artefacts.

But we start with the mathematics, which achieves its parsimonious elegance in a manner that is necessarily quite intricate.

The mathematics

At the core of the Magpie yield curve model is a parameterisation of a continuously-compounded instantaneous forward rate curve:

$$r(t) = y_0 + y_1 e^{-t/z_1} + \ldots + y_5 e^{-t/z_5}$$

And hence

$$Discount(s,t) =$$
$$\text{Exp}[y_0(s-t) + y_1 z_1 (e^{-t/z_1} - e^{-s/z_1}) + \ldots + y_5 z_5 (e^{-t/z_5} - e^{-s/z_5})]$$

This has two sets of parameters, the *z*'s, denominated in time, and the *y*'s, denominated in yield. Fitting both would be unstable, with multiple widely-separated local minima. Indeed, such an optimisation is closely related to the famously tricky problem of fitting the radioactive decay of multiple unknown elements. Instead we choose and fix the *z*'s to the values in the list on the right.

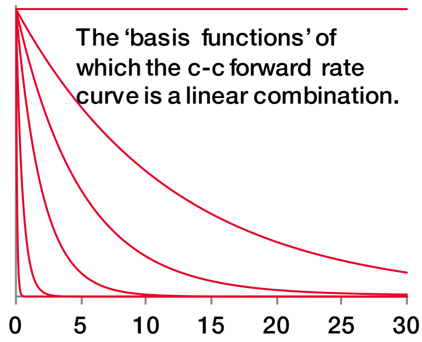
$$z_0 = +\infty$$
$$z_1 = 12 \text{ years}$$
$$z_2 = 5 \text{ years}$$
$$z_3 = 2 \text{ years}$$
$$z_4 = 6 \text{ months}$$
$$z_5 = 1 \text{ month}$$

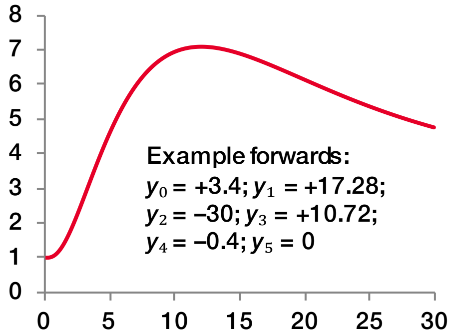
Observe that the *z*'s are well separated from each other, from infinity, and from zero, so each *y* does something, and does something different to the other *y*'s. And intuitively the *y*'s span the space of sensible smooth yield curves.

Indeed, the fitting problem is, in some sense, pre-diagonalised. Parameter y_0 is the FRA at infinity. With y_0 known then y_1 controls the shape of 10sLongs; with these two known then y_2 controls 5s10s; likewise y_3 and 2s5s; y_4 and most of sub-2Y; with y_5 setting the very short-term money-market detail.

This can leave a problem with the very short-dated part of the curve, which it might be thought could be 'over-fitted'. Consider a bond market in which there is a one-year bond, a two-year, and plenty of longer bonds. With these instruments the fastest decays (6 months and 1 month) are so fast that they are effectively the same: y_4 and y_5 do almost the same thing, and hence the fitting of y_4 and y_5 would be very sloppy. Indeed, a movement of 1bp in y_4 would very closely offset –5.188bp in y_5.

But if one were to ask a trader "what would be the yield of a 3-month government bond?", the trader would have an answer. The information exists: the trader would intuit the yield of a hypothetical government from elsewhere, perhaps from the shape of the repo curve. Likewise, the Magpie model accumulates such bright and shiny fragments of information from elsewhere. Conveniently, in both £ and €, there is a repo fixing: 1D, 1W, 2W, 3W, 1M, 2M, 3M, 6M, 9M, 1Y. These fixings show the shape of the short-dated curve, even in the absence of suitable government bonds.





So let there be two curves:

$$r^{\text{Govt}}(t) = y_0^{\text{Govt}} + y_1^{\text{Govt}} e^{-t/z_1} + \ldots + y_5^{\text{Govt}} e^{-t/z_5}$$
$$r^{\text{Repo}}(t) = y_0^{\text{Repo}} + y_1^{\text{Repo}} e^{-t/z_1} + \ldots + y_5^{\text{Repo}} e^{-t/z_5}$$

To fit the short-dated government y's and the long-dated repo y's the curves are 'linked':

$$\begin{aligned} y_1^{\text{Govt}} &= y_1^{\text{Repo}} \\ y_2^{\text{Govt}} &= y_2^{\text{Repo}} \\ y_3^{\text{Govt}} &= y_3^{\text{Repo}} \\ y_4^{\text{Govt}} &= y_4^{\text{Repo}} \\ y_5^{\text{Govt}} &= y_5^{\text{Repo}} \end{aligned}$$

So there are two curves, with common z's, different y_0s, and common other y's. Each of the non-y_0s has enough relevant steepness information, whether from gilts or repo, and there is enough overlap to fit two y_0s. Happiness.

■ A differentiatable error function

We still need an error function, which should, of course, be something like sum-square yield error. But using conventional yields would introduce an inelegance, and slowness, as they are computed by iteratively solving a 'conventional' polynomial.

We can do better:

$$Error = \sum_{\text{Tradeables}} \left(\frac{\text{Ln}\left(\dfrac{\text{Theoretical}[y_i]}{Market}\right)}{\dfrac{\partial \text{Theoretical}[y_i]}{\partial y_0} \times \dfrac{1}{\text{Theoretical}[y_i]}} \right)^2$$

The numerator is the price error, as a ratio. The denominator is a duration, though with respect to a parallel move in the forward rate curve, rather than wrt a change in an internal rate of return. Therefore this is something like a yield error.

Return to the equation for the discount factor from a settlement date s to payment at t. That discount is readily differentiated with respect to the y's, multiple times, and can then be summed for each bond. Thus we compute the first, second, and some of the third derivatives of the theoretical bond prices wrt the *y*'s. The error function can also be differentiated, which, with the instrument-specific derivatives already mentioned, gives the first and second derivatives of the error wrt to the *y*'s.

We are solving for the first derivatives to be zero, by Newton-Raphson. From a fixed rubbish starting guess (the curve flat at 1bp per day), it has been taking two to four steps to reach machine precision.

Indeed, for a single-cashflow instrument the contribution to the error simplifies to precisely sum-square yield error (of course quoted continuously compounded). Further, the double derivatives of the error contribution of a single-cashflow instrument, with respect to the *y*'s, are independent of the *y*'s: the error is precisely parabolic. Thus for instruments with constant duration (wrt yield) Newton-Raphson finds the optimum in one step — though a second step might be wanted to polish away the numerical errors caused by finite machine precision.

Other models

Readers might wish to compare Magpie to Filtering the interest rate curve: The MENIR framework, which uses a Kalman filter to reduce instability in fitted parameters.

Other models include those by or inspired by Nelson-Siegel, which generally have the forward rates as a product between a polynomial and an exponential decay term. The author prefers the sum of exponents, as there is a wider class of natural processes generating such.

Alternatively, if one is willing to use history as well as an instantaneous snapshot of prices, PCA-based techniques can be useful. But these can provide only limited or indirect help in valuing newly issued securities.

Further, Magpie is the only model, of those known to the author, that allows information to be smoothly moved between 'similar' curves.

High-speed mathematics

If using this for trading at very high frequency — not yet at Société Générale — the optimisation can be made even faster. We already have the second derivatives of the error wrt to the *y*'s. Also compute, for each instrument's contribution to the error, more double derivatives, once with respect to that instrument's market price, and once wrt the *y*'s.

What is this? Vary the market price of bond *j* by ∂M_j, multiply by the freshly computed $\partial^2 E/\partial y_i \partial M_j$ to get the change in $\partial E/\partial y_i$. At the *y*-minimum of the error, $\partial E/\partial y_i$ should be zero. So pre-multiply by the inverse of second derivatives of the error wrt to *y* twice, and by –1, to get the Newton-Raphson change in the *y*'s, computed analytically. More formally:

$$\begin{pmatrix} \partial y_0/\partial M_j \\ \vdots \\ \partial y_5/\partial M_j \end{pmatrix} = -\begin{pmatrix} \partial^2 E/\partial y_0 \partial y_0 & \cdots & \partial^2 E/\partial y_0 \partial y_5 \\ \vdots & \ddots & \vdots \\ \partial^2 E/\partial y_5 \partial y_0 & \cdots & \partial^2 E/\partial y_5 \partial y_5 \end{pmatrix}^{-1} \begin{pmatrix} \partial^2 E/\partial M_j \partial y_0 \\ \vdots \\ \partial^2 E/\partial M_j \partial y_5 \end{pmatrix}$$

That is, the model gives analytical values for the partial differential of the optimised parameters with respect to each input price. Of course, these can be multiplied by the $\partial \text{Theory}_j / \partial y_i$ to give changes in theoretical values. Yes, the model is so stable that re-optimisation can be done with matrix multiplication.

The output

The next few pages show example output. This takes the form of full-page charts, packed with information.

The first chart is plotted against an *x* axis showing dates, albeit gently distorted. The thin black line shows the instantaneous forward rates, quoted at the same frequency as the bonds, so semi-annual in £. These forward rates start at 37bp, stay below 50bp until end-2013, then climb steeply until reaching a maximum above the top of the chart at 4.58% (statistic given in text bottom-right), before descending towards a limit at infinity of 2.67%.

These forward rates imply zero-coupon rates, again quoted semi-annual, shown as a thick black line.

These zero-coupon rates imply a par curve, shown as a thick grey line, and also imply ammortiser yields (that is, all coupon and no principal, or equivalently, yields of bonds with gigantic coupons).

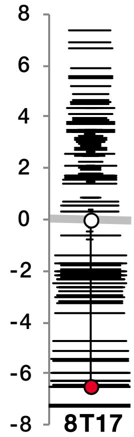
In red are the market yields of the bonds. In white are their theoretical yields, that is, the yield implied by discounting each cashflow at its zero-coupon yield.

Unsurprisingly, some bonds are cheap and some are dear. That is part of the purpose of the rigidity of this yield curve model: it doesn't fit all the instruments, thereby revealing cheap-dear.

Observe that the gilts from late 2019 to 2022 are all cheap. To fit these exactly the forward rate curve would have to be steeper to 2022. But the 2025 to 2030 gilts are all dear: to fit these exactly the forward rates from Mar 2022 to 2030 need to be lower. This steeper and then much lower implies a very sharp turn in the forward rates; too sharp for this model. Indeed, too sharp to be plausible. Thus the model's assumption of very smooth forwards implies that the 7½Y to 10Y gilts are relatively cheap sets of cashflows; that the 13Y to 19Y are relatively dear; and that some longer bonds are relatively cheap.

The next page shows the spot instruments plotted locally against the natural *x* axis against which to show yields, Macaulay-Weil convexity ÷ Macaulay-Weil duration (explanation thereof). Locally? The marking of the *x* axis, and the ordering of the instruments, is convexity÷duration. Over a small range of values the axis is about linear in this. But the axis has been distorted (as was the previous chart's date axis) by an exponent of 0.6, to expand the short-dated part of the curve (which has a higher information density and instrument density), and compress the long-dated part (which has less). All this complication is to find the best combination of fixed-income pedantry, and of clarity of presentation.

The next three charts zoom in on parts of the curve, and show the history of the relative value. The final chart shows just the history of the relative value. One explanation, that of the last, will suffice for all four.



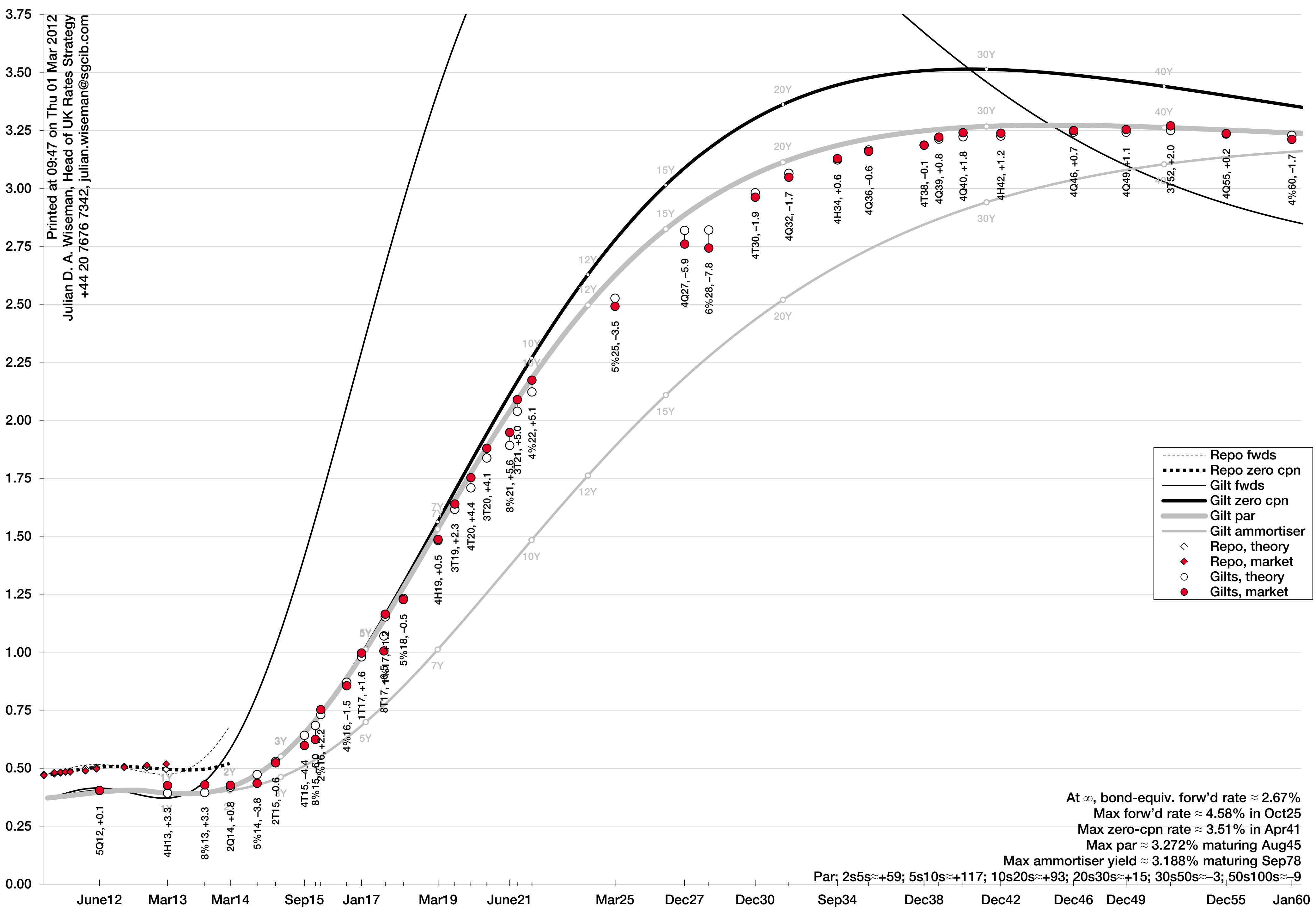
The *y* axis zero is fair value, for each bond marked with the white circle. The red circle is the market yield relative to this, so positive is cheap. So the 8T17, also pictured right, is 6.5bp dear. Six months ago it was slightly cheap, but now is near its recent record dearness.

There is a technical division of history into 'old' and not. In late September 2011 the UK DMO created a new gilt, the 3¾ July 2052. Since issue the 3T52 has traded a little cheap. The creation of this cheap security would therefore have shifted slightly upwards the fair yields for other long gilts. Even if their market yields hadn't moved, their apparent relative value would have had a small jump, slightly dearer. To avoid confusion with this artefact of a new issue, older history is shown pale grey. Decisions about the demotion of history must and will be made non-automatically.

Conclusion

Magpie gives a stable means of assessing micro relative value, presented clearly.

It will be of use to investors willing to engage in small switches and barbells.



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3.75
3.50
3.25
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
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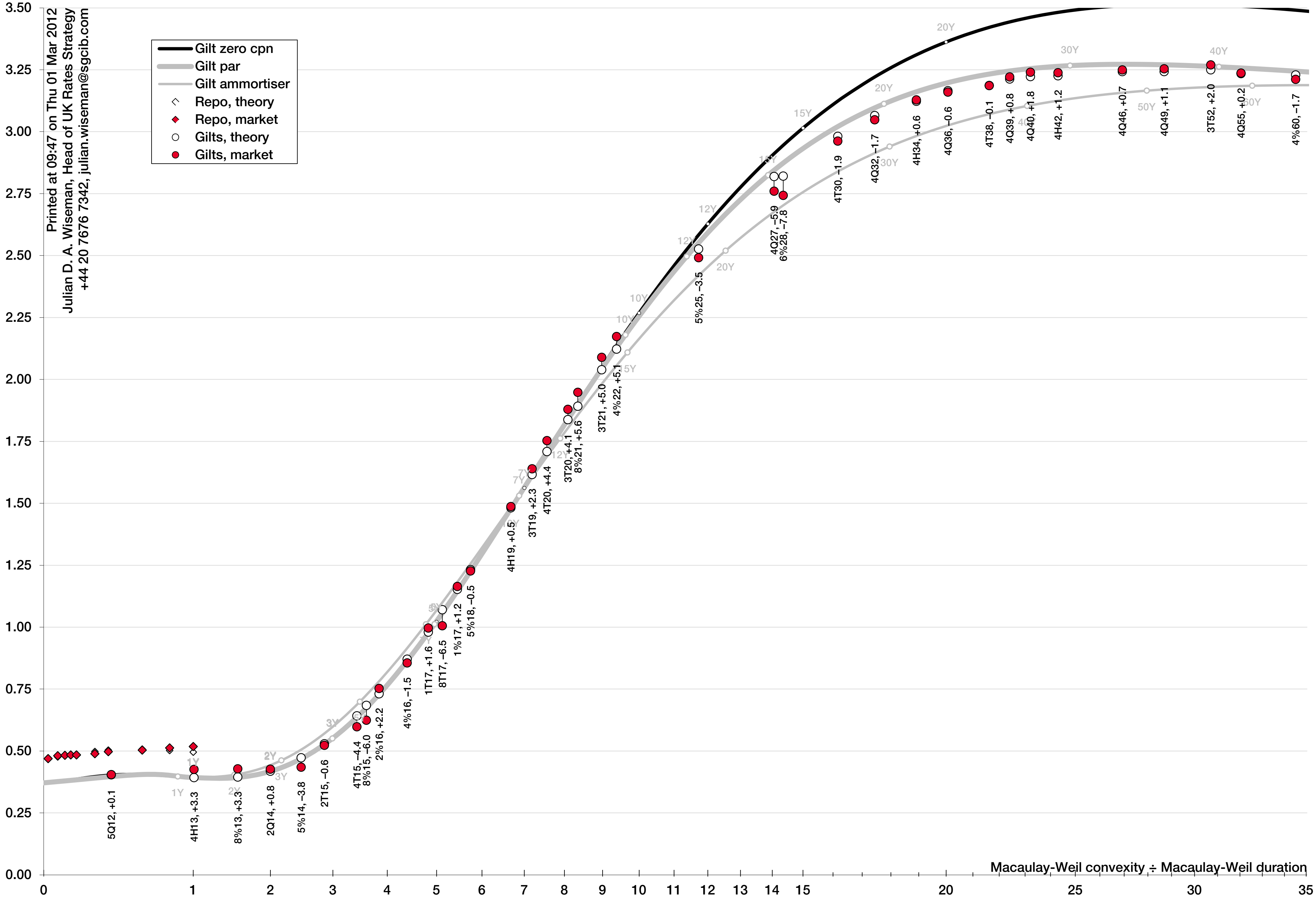
June12
Mar13
Mar14
Sep15
Jan17
Mar19
June21
Mar25
Dec27
Dec30
Sep34
Dec38
Dec42
Dec46
Dec49
Dec55
Jan60
Repo fwds
Repo zero cpn
Gilt fwds
Gilt zero cpn
Gilt par
Gilt ammortiser
Repo, theory
Repo, market
Gilts, theory
Gilts, market
5Q12, +0.1
4H13, +3.3
8%13, +3.3
2Q14, +0.8
5%14, −3.8
2T15, −0.6
4T15, −4.4
4%16, −1.5
1T17, +1.6
5%18, −0.5
4H19, +0.5
3T19, +2.3
4T20, +4.4
3T20, +4.1
4%22, +5.1
5%25, −3.5
4Q27, −5.9
6%28, −7.8
4T30, −1.9
4Q32, −1.7
4H34, +0.6
4Q36, −0.6
4T38, −0.1
4Q39, +0.8
4Q40, +1.8
4H42, +1.2
4Q46, +0.7
4Q49, +1.1
4Q55, +0.2
4%60, −1.7
At ∞, bond-equiv. forw'd rate ≈ 2.67%
Max forw'd rate ≈ 4.58% in Oct25
Max zero-cpn rate ≈ 3.51% in Apr41
Max par ≈ 3.272% maturing Aug45
Max ammortiser yield ≈ 3.188% maturing Sep78
Par; 2s5s≈+59; 5s10s≈+117; 10s20s≈+93; 20s30s≈+15; 30s50s≈−3; 50s100s≈−9



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Gilt zero cpn
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Macaulay-Weil convexity ÷ Macaulay-Weil duration
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2T15, −0.6
4T15, −4.4
8%15, −6.0
2%16, +2.2
4%16, −1.5
1T17, +1.6
8T17, −6.5
1%17, +1.2
5%18, −0.5
4H19, +0.5
3T19, +2.3
4T20, +4.4
3T20, +4.1
8%21, +5.6
3T21, +5.0
4%22, +5.1
5%25, −3.5
4Q27, −5.9
6%28, −7.8
4T30, −1.9
4Q32, −1.7
4H34, +0.6
4Q36, −0.6
4T38, −0.1
4Q39, +0.8
4Q40, +1.8
4H42, +1.2
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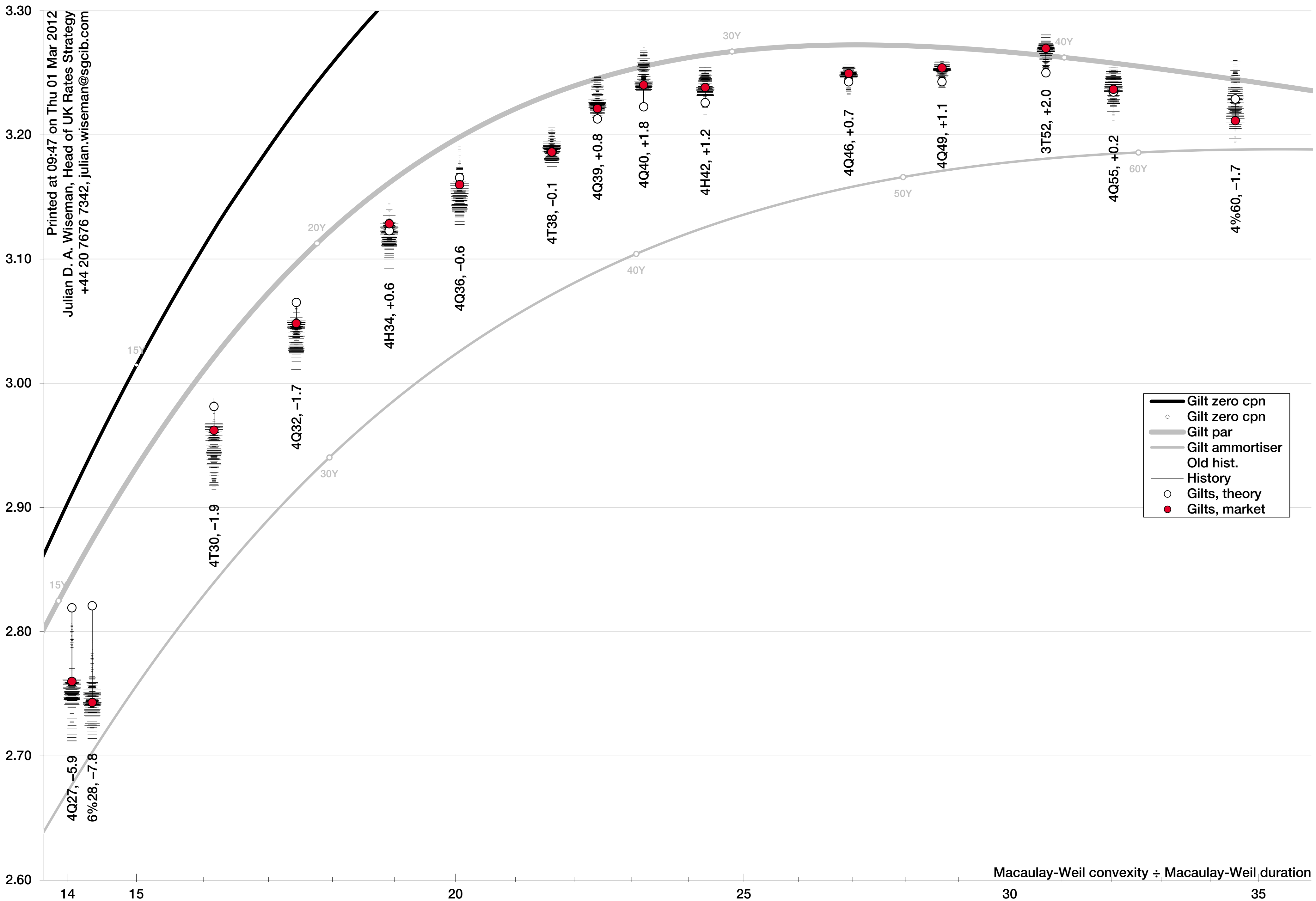

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2.70
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35
Macaulay-Weil convexity ÷ Macaulay-Weil duration
15Y
20Y
30Y
40Y
50Y
60Y
4Q27, −5.9
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Gilt zero cpn
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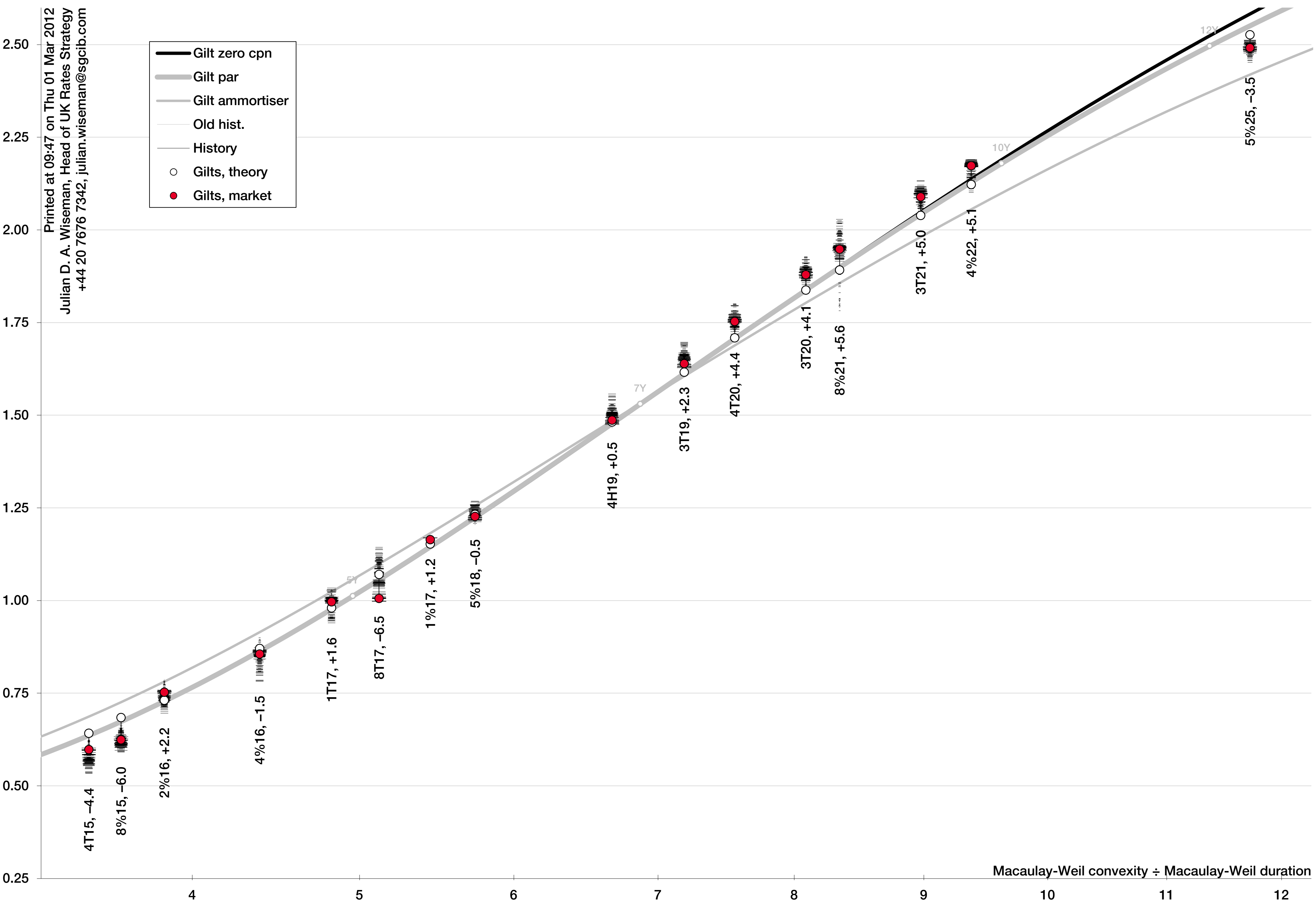
Gilt ammortiser
Old hist.
History
Gilts, theory
Gilts, market



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8%21, +5.6
3T21, +5.0
4%22, +5.1
5%25, −3.5
5Y
7Y
10Y
12Y
2.50
2.25
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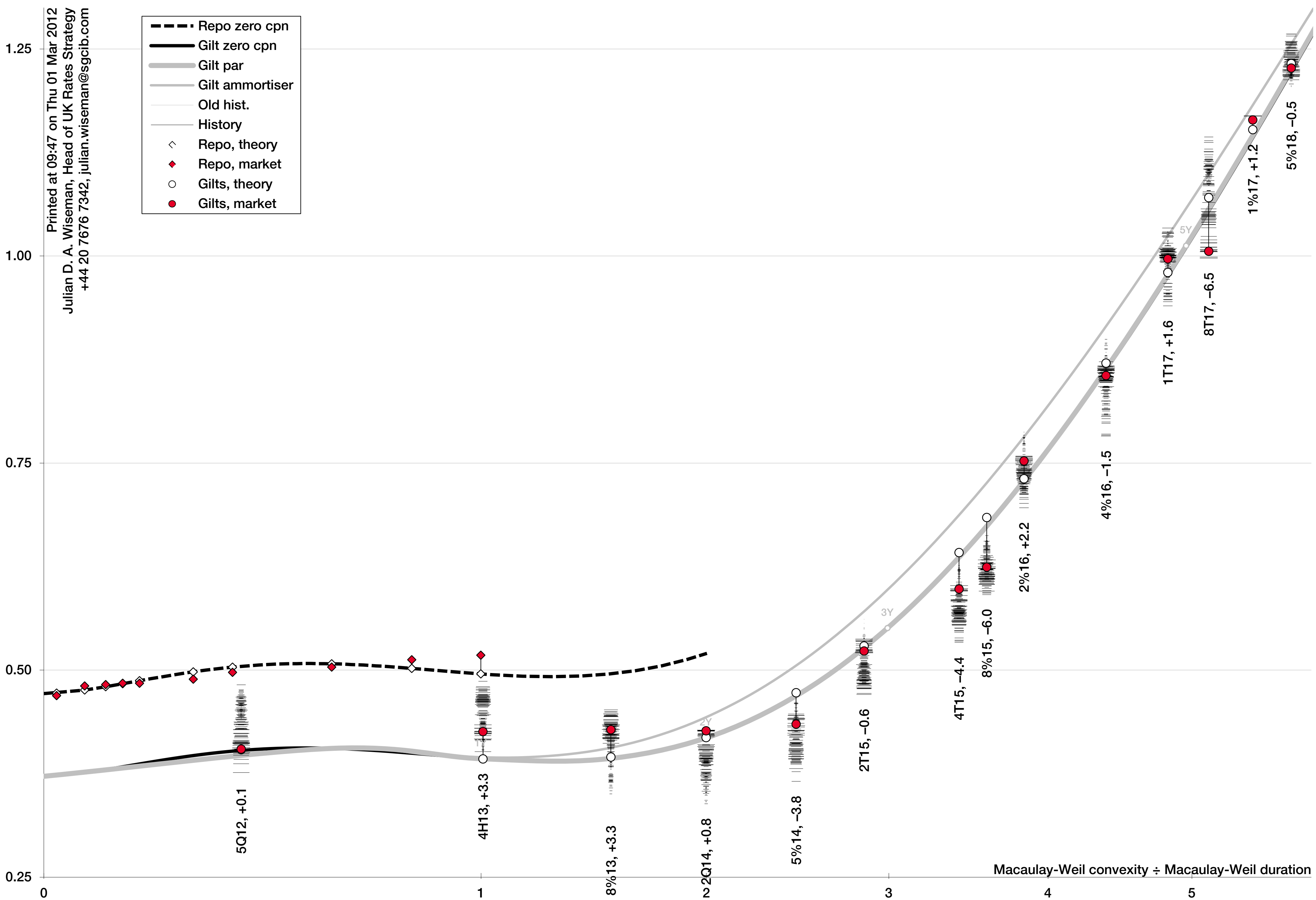
Macaulay-Weil convexity ÷ Macaulay-Weil duration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1.25
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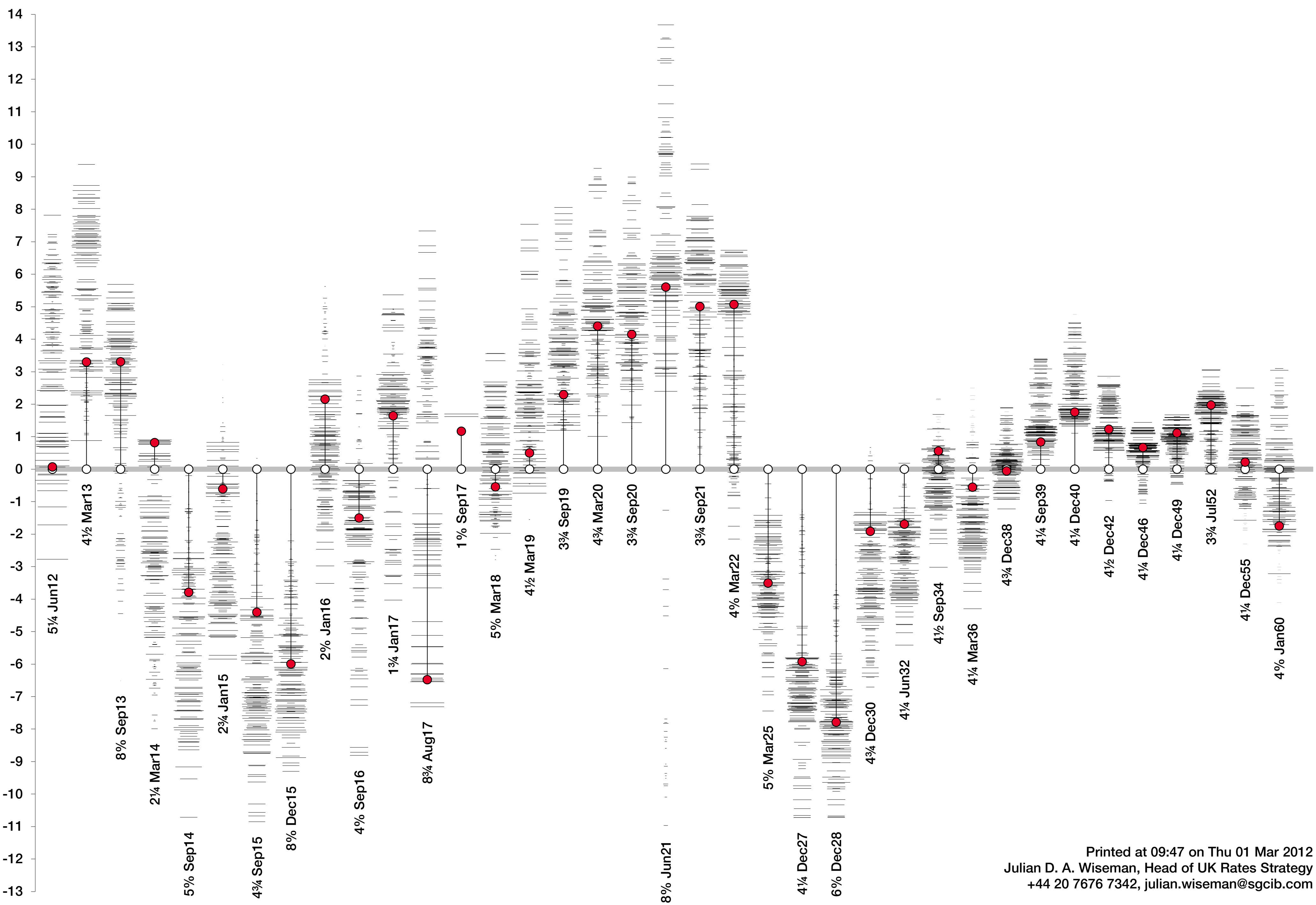
2T15, −0.6
4T15, −4.4
8%15, −6.0
2%16, +2.2
4%16, −1.5
1T17, +1.6
8T17, −6.5
1%17, +1.2
5%18, −0.5
3Y
5Y



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